Dynamical models explaining social balance

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Heider's theory on social balance (1946)

Consider n individuals and let X_{ij} denote the real-valued reputation i has about j.

$X_{ij} > 0$ for friend; < 0 for enemy, and X_{ii} is self-esteem.

$n \times n$ matrix X is called the reputation matrix.

Def: Network is socially balanced if:

$$X_{ij}X_{ik}X_{kj} > 0, \ \forall i, j, k$$

A friend of a friend is a friend; enemy of an enemy is a friend.
A friend of an enemy is an enemy; enemy of a friend is an enemy.

Heider's theory on social balance (1946)

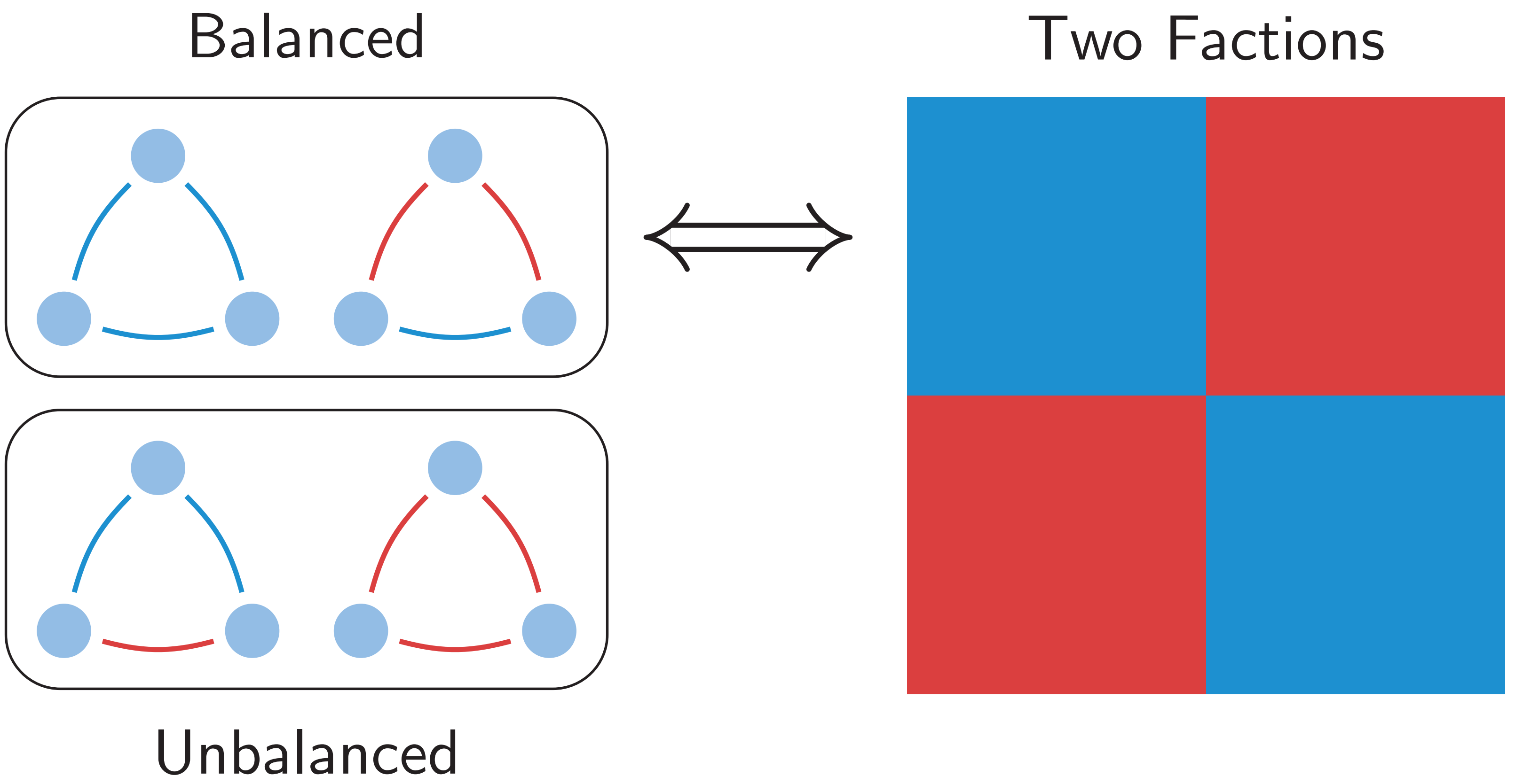
Structure Thm (Cartwright & Harary, 1956): Social balance iff, up to permutation, X takes one of two block forms:

$$X = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Thus, at most 2 factions in the network.
Individuals in same faction are friends.
Individuals in different factions are enemies.

Examples: World Peace (?), WW II (allied powers/axis powers), Cold War (NATO/Warsaw pact), democracy (majority/opposition),...

Heider's theory on social balance (1946)



Heider's theory on social balance (1946)

- Heider's theory is static, but based on dynamic ideas. So, there is a need to develop dynamic models which exhibit ("converge to") social balance.

- Kulakowsky etal in 2005 numerically studied:

$$\dot{X} = X^2, \text{ i.e. } \dot{X}_{ij} = \sum_k X_{ik} X_{kj} \tag{1}$$

  and observed emergence of social balance.
  Model interpretation: i updates opinion about j, based on gossip from k: the product of
  (i) i's opinion about gossiping partner k, and
  (ii) what k thinks about j.
  Note: Self-esteem X_{ii} has a positive effect, X_{ii}^2.

Dynamic models of social balance

Technical issue: Finite-time blow up, even if $n = 1$:

$$\dot{x} = x^2, \;\; x(0) = x_0,$$

$$\text{Solution: } x(t) = \frac{x_0}{1 - x_0 t},$$

defined only for $t \in [0, 1/x_0)$ if $x_0 > 0$.

To remedy this, we shall consider social balance in a normalized sense:

$$\lim_{t \to \bar{t}} \frac{X(t)}{|X(t)|_F}, \;\; |X|_F := \left(\text{tr}(XX^T)\right)^{1/2} \text{ Frobenius norm}$$

where $\bar{t}$ could be finite, in case $X(t)$ blows up in finite time. What matters for social balance are *the signs* of the entries $X_{ij}(t)$, not their magnitude.

Dynamic models of social balance

Strogatz etal in 2011 proved that for (1), if $X(0) = X(0)^T$:

$$\lim_{t \to 1/\lambda_1} \frac{X(t)}{|X(t)|_F} = U_1 U_1^T$$

provided eigenvalues of $X(0)$: $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1 > 0$, and $X(0)U_1 = \lambda_1 U_1$.

Thus, if, up to permutation,:

$$U_1 = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + \\ - \end{pmatrix} \implies U_1 U_1^T = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

then social balance is achieved in finite time.

Dynamic models of social balance

Pf: Diagonalize $X(0) = X(0)^T$:

$$X(0) = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} U^T, \;\; UU^T = I_n$$

Then $X(t) = UD(t)U^T$ solves (1), where diagonal $D(t)$:

$$\dot{D}_j = D_j^2, D_j(0) = \lambda_j \text{ has sol. } D_j(t) = \frac{\lambda_j}{1 - \lambda_j t}$$

Finite escape time $1/\lambda_1$, and

$$\lim_{t \to 1/\lambda_1} \frac{X(t)}{|X(t)|_F} = U \left(\lim_{t \to 1/\lambda_1} \frac{D(t)}{|D(t)|_F} \right) U^T = U e_1 e_1^T U^T = U_1 U_1^T$$

Can symmetry assumption $X(0) = X^T(0)$ be relaxed?

Special, but restrictive case of normal $X(0)$:

$$X(0)X^T(0) = X^T(0)X(0),$$

Block-diagonalize $X(0)$:

$$X(0) = U \begin{pmatrix} A_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & A_k & 0 & \dots & 0 \\ 0 & \dots & 0 & B_1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & B_l \end{pmatrix} U^T,\ UU^T = I_n$$

$$A_i = (a_i),\ B_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix},\ \beta_j \neq 0$$

$$\dot{B} = B^2,\ B(0) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\ \beta \neq 0$$

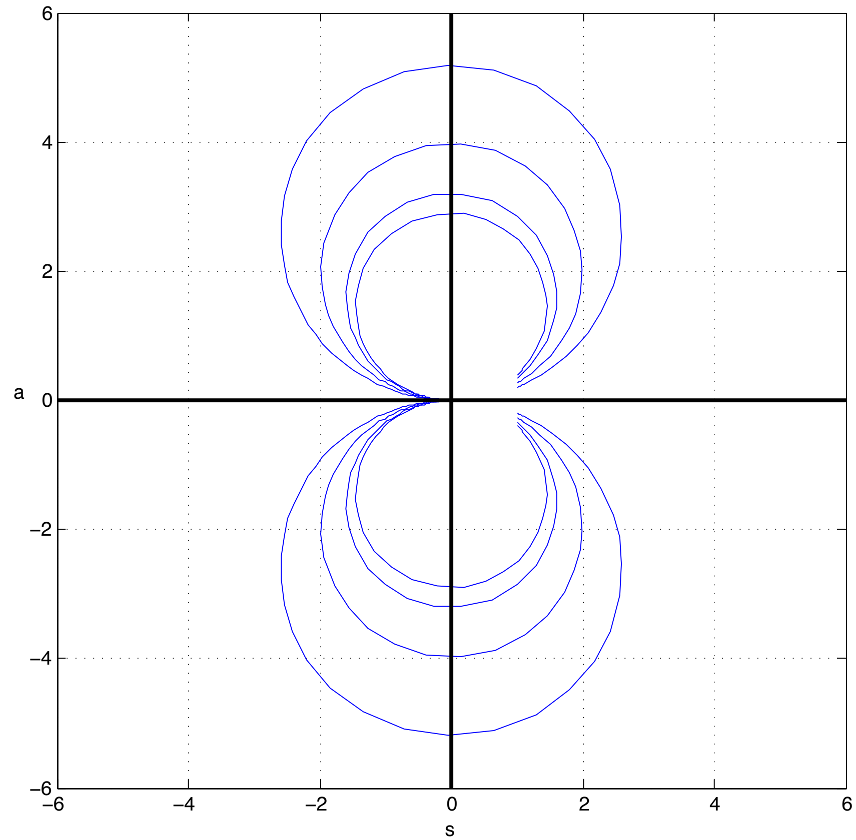
Prop $\lim_{t \to +\infty} \frac{B(t)}{|B(t)|_F} = -\frac{\sqrt{2}}{2} I_2$.

Pf: Split $B = S + A$ into symmetric $S = S^T$ and anti-symmetric $A = -A^T$ parts:

$$\begin{aligned} \dot{S} &= S^2 + A^2,\ S(0) = \alpha I_2 \\ \dot{A} &= AS + SA,\ A(0) = \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

$S(t)$ remains multiple of I_2, and hence it suffices to consider the planar system:

$$\begin{aligned} \dot{s} &= s^2 - a^2,\ \ s(0) = \alpha \\ \dot{a} &= 2as,\ \ a(0) = \beta \end{aligned}$$



Circular orbits: $s^2 + (a-c)^2 = c^2$, and $\lim_{t\to+\infty} \frac{a(t)}{s(t)} = 0$.

$\Longrightarrow \lim_{t\to+\infty} \frac{B(t)}{|B(t)|_F} = \lim_{t\to+\infty} \frac{S(t)+A(t)}{|S(t)+A(t)|_F} = -\frac{\sqrt{2}}{2} I_2.$

Social balance when $X(0)$ normal

$$X(t) = U \begin{pmatrix} a_1/(1-a_1t) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & a_k/(1-a_kt) & 0 & \dots & 0 \\ 0 & \dots & 0 & B_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & B_l(t) \end{pmatrix} U^T$$

Thm Let $X(0)$ be normal and $a_1 > a_2 \geq \cdots \geq a_k$ with $a_1 > 0$ and $X(0)U_1 = a_1U_1$. Then

$$\lim_{t \to 1/a_1} \frac{X(t)}{|X(t)|_F} = U_1U_1^T$$

Further relaxation of $X(0)$?

Put $X(0)$ in Jordan canonical form:

$$X(0) = T \begin{pmatrix} A_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & A_k & 0 & \dots & 0 \\ 0 & \dots & 0 & B_1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & B_l \end{pmatrix} T^{-1},\ TT^{-1} = I_n$$

$$A_i = \begin{pmatrix} a_i & 1 & \dots & 0 \\ 0 & a_i & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_i \end{pmatrix},\ B_j = \begin{pmatrix} C_i & I_2 & \dots & 0 \\ 0 & C_i & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_i \end{pmatrix},$$

$$C_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix},\ \beta_j \neq 0$$

$$\Longrightarrow \text{ Solution: } X(t) = T \begin{pmatrix} A_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_l(t) \end{pmatrix} T^{-1}$$

Thm Let $a_1 > a_2 \geq \cdots \geq a_k$ with $a_1 > 0$ simple, and $X(0)T_1 = a_1 T_1$, and $V_1^T X(0) = a_1 V_1^T$. Then

$$\lim_{t \to 1/a_1} \frac{X(t)}{|X(t)|_F} = \frac{T_1 V_1^T}{|T_1 V_1^T|_F} = \frac{T_1 V_1^T}{||T_1||.||V_1||}$$

Remark: Although convergence to rank 1 matrix, there is no guaranteed social balance! Up to permutation:

$$T_1 V_1^T = \begin{pmatrix} + \\ \hline - \end{pmatrix} \begin{pmatrix} + & - & | & + & - \end{pmatrix} = \begin{pmatrix} + & - & + & - \\ - & + & - & + \end{pmatrix}$$

A second social model

$$\dot{X} = XX^T, \text{ i.e. } \dot{X}_{ij} = \sum_k X_{ik} X_{jk} \tag{2}$$

Thus, i updates opinion on j, based on gossip with j about k: product of
(i) i's opinion about k, and
(ii) j's opinion about k. (reversed!)
Def: Network is socially balanced if:

$$X_{ij} X_{ik} X_{jk} > 0, \ \forall i, j, k$$

Equivalent to previous definition, since $(i, j, k) \to (i, k, j)$, so Structure Thm remains valid: Social balance iff

$$X = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Assume $X(0)$ normal, block-diagonalize with $UU^T = I_n$:

$$X(t) = U \begin{pmatrix} a_1/(1-a_1 t) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & a_k/(1-a_k t) & 0 & \dots & 0 \\ 0 & \dots & 0 & B_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & B_l(t) \end{pmatrix} U^T$$

where each $B_j(t)$ satisfies:

$$\dot{B} = BB^T,\ B(0) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\ \beta \neq 0$$

Split $B(t) = S(t) + A(t),\ S(t) = S^T(t),\ A(t) = -A^T(t)$:

$$\begin{aligned} \dot{S} &= (S+A)(S-A),\ S(0) = \alpha I_2 \\ \dot{A} &= 0,\ A(0) = \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \implies A(t) = A(0) \end{aligned}$$

$$\dot{S} = (S+A(0))(S-A(0)),\ S(0) = \alpha I_2,\ A(0) = \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$S(t)$ remains multiple of I_2, so consider diagonal entry:

$$\dot{s} = s^2 + \beta^2,\ s(0) = \alpha$$

Scalar Riccati equation; solution:

$$s(t) = \beta \tan(\beta t + \phi),\ \ t \in [0, \bar{t})$$

$$\phi := \arctan(\alpha/\beta),\ \ \bar{t} := \frac{\pi}{2|\beta|} - \frac{\phi}{\beta} > 0 \text{ (blow-up time)}$$

Then,

$$\lim_{t \to \bar{t}} \frac{B(t)}{|B(t)|_F} = \lim_{t \to \bar{t}} \frac{S(t) + A(t)}{|S(t) + A(t)|_F} = \frac{\sqrt{2}}{2} I_2$$

$\dot{X} = XX^T$: Social balance or not when $X(0)$ normal

Thm Let $X(0)$ be normal and $a_1 > a_2 \geq \cdots \geq a_k$ and $X(0)U_1 = a_1 U_1$. Let

$$\bar{t}_1 < \bar{t}_2 \leq \cdots \leq \bar{t}_k, \;\; \bar{t}_j := \frac{\pi}{2|\beta_j|} - \frac{\phi}{\beta_j}, \; \phi_j = \arctan(\alpha_j/\beta_j)$$

and $\tilde{U}_1$ the 2 columns of U that correspond to B_1.

If $0 < 1/a_1 < \bar{t}_1$, then $\lim_{t \to 1/a_1} \frac{X(t)}{|X(t)|_F} = U_1 U_1^T$ (**social balance**).

If $\bar{t}_1 < 1/a_1$, then $\lim_{t \to \bar{t}_1} \frac{X(t)}{|X(t)|_F} = \frac{\sqrt{2}}{2} \tilde{U}_1 \tilde{U}_1^T$ (**no guaranteed social balance, see next slide**).

- If $X(0)$ has real e-value $a > 0$, or e-value pair $\alpha \pm j\beta$ with $\beta \neq 0$, then always blow-up in finite time.

- Competition between the largest real, positive eig-value, and complex pair of eig-values determines blow-up time:
(i) If real, positive one wins (i.e. blows up first), then social balance.
(ii) If complex pair wins, then social balance cannot be guaranteed.

- If complex pair wins, convergence to rank 2 matrix:

$$\frac{\sqrt{2}}{2}\tilde{U}_1\tilde{U}_1^T = \frac{\sqrt{2}}{2}\begin{pmatrix} + & + \\ + & - \\ - & + \\ - & - \end{pmatrix}\begin{pmatrix} + & + & - & - \\ + & - & + & - \end{pmatrix} = \begin{pmatrix} + & ? & ? & - \\ ? & + & - & ? \\ ? & - & + & ? \\ - & ? & ? & + \end{pmatrix}$$

Relaxing normality condition on $X(0)$ for $\dot{X} = XX^T$.

Symm. and anti-symm. parts satisfy:

$$\begin{aligned} \dot{S} &= (S+A)(S-A) \\ \dot{A} &= 0 \Longrightarrow A(t) = A_0 \end{aligned}$$

S-eqn. is a matrix RDE. Eliminate linear terms:

$$\widehat{S}(t) = \mathrm{e}^{-tA_0}\, S(t)\, \mathrm{e}^{tA_0}$$

$$\Longrightarrow \quad \dot{\widehat{S}} = \widehat{S}^2 - A_0^2,\ \widehat{S}(0) = S_0$$

Block-diagonalize $-A_0^2$ with $VV^T = I_n$:

$$-A_0^2 = V^T \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \omega_1^2 I_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_k^2 I_2 \end{pmatrix} V^T,\ \text{all } \omega_j > 0$$

Then setting $\tilde{S} = V^T \hat{S} V$ and $\tilde{S}_0 = V^T S_0 V$:

$$\dot{\tilde{S}} = \tilde{S}^2 + D^2,\ \tilde{S}(0) = \tilde{S}_0,\ D^2 := \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D}^2 \end{pmatrix},\ \tilde{D} := \begin{pmatrix} \omega_1 I_2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_k I_2 \end{pmatrix}$$

This is a matrix RDE, whose solution follows from:

$$\begin{pmatrix} \dot{P} \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ D^2 & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix},\quad \begin{pmatrix} P(0) \\ Q(0) \end{pmatrix} = \begin{pmatrix} I_n \\ \tilde{S}_0 \end{pmatrix}$$

Then:

$$\tilde{S}(t) = Q(t) P^{-1}(t),\ \left(\frac{P(t)}{Q(t)}\right) = \left(\frac{\begin{pmatrix} I_{n-2k} & 0 \\ 0 & c \end{pmatrix} - \begin{pmatrix} t I_{n-2k} & 0 \\ 0 & \tilde{D}^{-1} s \end{pmatrix} \tilde{S}_0}{\begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} s \end{pmatrix} + \begin{pmatrix} I_{n-2k} & 0 \\ 0 & c \end{pmatrix} \tilde{S}_0}\right),$$

$$s := \begin{pmatrix} \sin(\omega_1 t) I_2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sin(\omega_k t) I_2 \end{pmatrix} \text{ (and } c \to \cos\text{)}$$

Social balance for $\dot{X} = XX^T$ with generic $X(0)$

Thm Assume $A_0 \neq 0$, then $X(t)$ blows up at some finite $\bar{t}$.

Moreover, there is a dense set of $X(0)$ such that the limit

$$\lim_{t \to \bar{t}} \frac{X(t)}{|X(t)|_F} = \frac{(\mathrm{e}^{\bar{t}A_0} Vu)(\mathrm{e}^{\bar{t}A_0} Vu)^T}{|uu^T|_F}$$

for some vector $u \neq 0$.

Since the limit is a rank 1 matrix of the form yy^T for some vector y, social balanced is achieved.

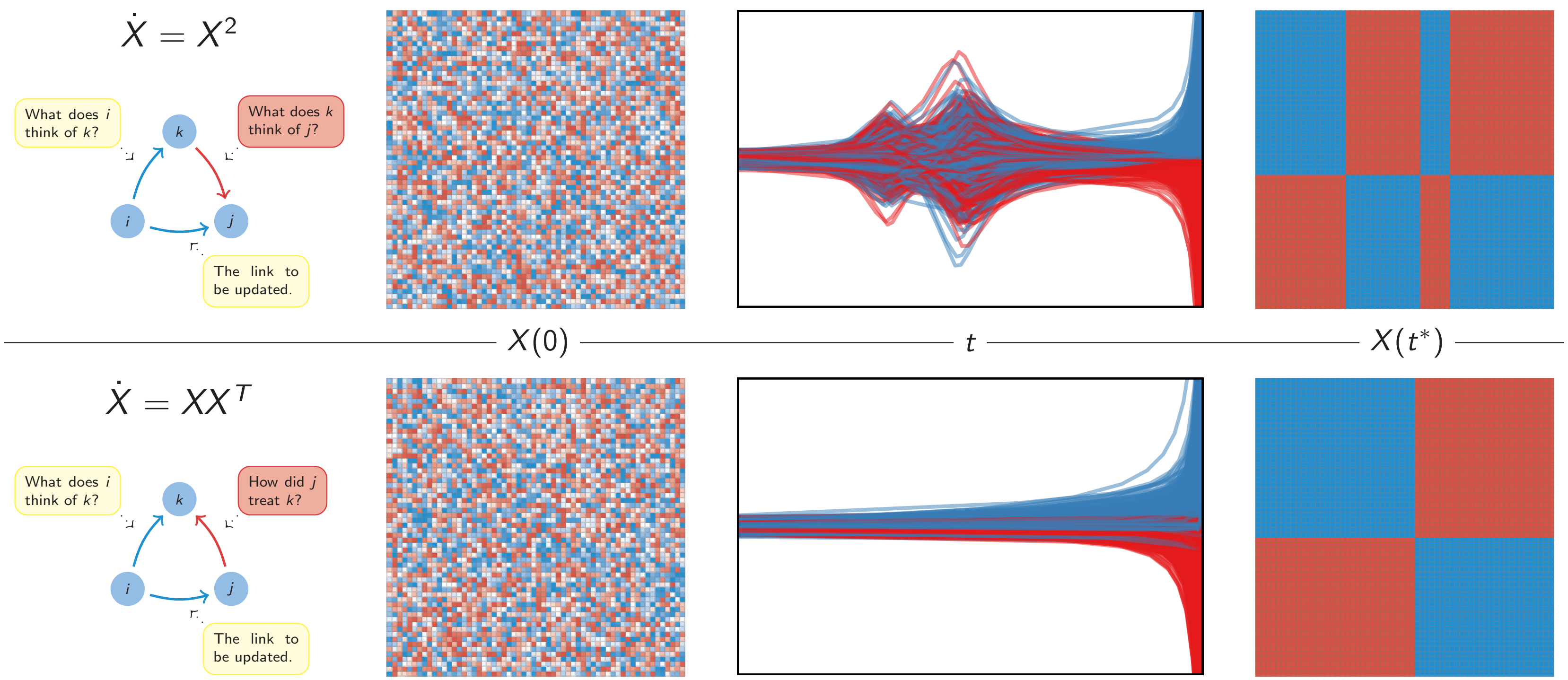
Conclusions

1. For $\dot{X} = X^2$, social balance is achieved if $X(0)$ is normal, but generically not achieved if it is not normal.

2. For $\dot{X} = XX^T$, social balance is generically achieved for non-normal $X(0)$.

   Nevertheless, we proved that it is *sometimes* achieved for normal $X(0)$ (namely when blow-up is due to a simple, real and positive eigenvalue of $X(0)$).

Conclusions in 1 picture



Thank you!

More details in:

Dynamical Models Explaining Social Balance and Evolution of Cooperation,
V. Traag, P. Van Dooren, and P. De Leenheer,
PLOS ONE 8 (4), April 2013.
(also arXiv:submit/0522501).

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